HW #7 SOLUTIONS

You should use these two eqns to solve problems. Get familiar with those equations.

$$\Sigma Q \delta_P = \int_{x=0}^{x=L} M_Q \frac{M_P dx}{EI}$$

$$\Sigma Q_M \theta_P = \int_{x=0}^{x=L} M_Q \frac{M_P dx}{EI}$$

where $Q \sim$ dummy load and its reactions

$\delta_P \sim$ actual displacement or component of displacement in direction of dummy load produced by real loads

$\theta_P \sim$ actual slope in direction of dummy load produced by real loads

$M_Q \sim$ moment produced by dummy load

$M_P \sim$ moment produced by real loads

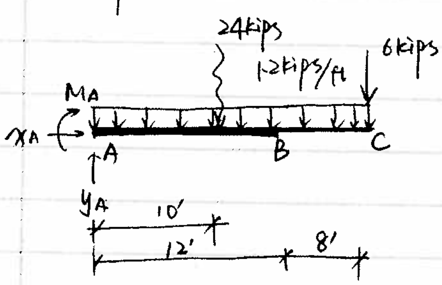
$E \sim$ modulus of elasticity

$I \sim$ moment of inertia of beams cross section with respect to an axis through centroid

P10.18 Compute the deflection and slope at point C. Given $E = 29{,}000\,\text{kips/in}^2$, $I = 1200\,\text{in}^4$.

1) Compute external reactions:

$\uparrow \curvearrowright \oplus$ **P system**

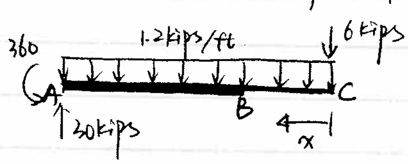


$\Sigma F(x) = A_x = 0$

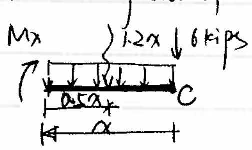
$\Sigma M_A = M_A + 24 \times 10 + 6 \times 20 = 0 \quad M_A = -360\,\text{ft-kips} \circlearrowleft$

$\Sigma F_y = y_A - 24 - 6 = 0 \quad y_A = 30\,\text{kips} \uparrow$

Draw FBD, show original point and direction, write internal force M in terms of x.



Take a segment of the beam:



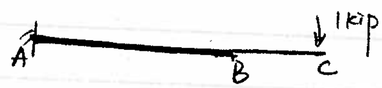
$\curvearrowright \oplus$

$\Sigma M_x = M_x + 1.2x(0.5x) + 6x = 0$

$M_x = -0.6x^2 - 6x \quad (0 \le x \le 20')$

2) **Q system**

If we want to compute vertical displacement at C, we should apply dummy force 1 kip at C along vertical direction.



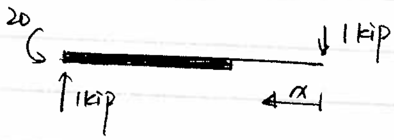
Compute external reactions based on this loading condition, then write M in terms of x.

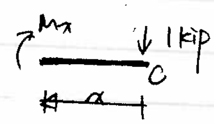
$\Sigma F(x) = x_A = 0$

$\Sigma M_A = M_A + 1 \cdot 20 = 0 \quad M_A = -20$ ft-kips ↻

$\Sigma F_y = y_A - 1 = 0 \quad y_A = 1$ kip ↑

FBD:





$\Sigma M_x = M_x + x = 0 \quad M_x = -x \; (0 \le x \le 20')$

Because of discontinuity in I value, we should divide this beam into 2 parts, $0 \le x \le 8'$ & $8' \le x \le 20'$

3) Compute δ_{CV}.

$$\because \Sigma Q \delta_P = \int M_Q \frac{M_P dx}{EI}$$

$$1 \text{ kip} \cdot \delta_{CV} = \int_0^8 \frac{(-x)(-0.6x^2 - 6x)dx}{EI} + \int_8^{20} \frac{(-x)(-0.6x^2 - 6x)dx}{E(2I)}$$

$$= \int_0^8 \frac{(0.6x^3 + 6x^2)dx}{EI} + \int_8^{20} \frac{(0.6x^3 + 6x^2)dx}{2EI}$$

$$= \frac{1}{EI}(0.15x^4 + 2x^3)\Big|_0^8 + \frac{1}{2EI}(0.15x^4 + 2x^3)\Big|_8^{20}$$

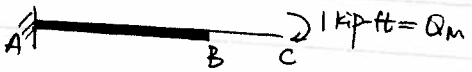
$$= \frac{1}{EI}(0.15 \times 8^4 + 2 \times 8^3) + \frac{1}{2EI}(0.15 \times 20^4 + 2 \times 20^3 - 0.15 \times 8^4 - 2 \times 8^3)$$

$$= \frac{1638.4}{EI} + \frac{38361.6}{2EI}$$

$$= \frac{1638.4 \times (12^3)}{29{,}000 \times 1200} + \frac{38361.6 \times (12^3)}{2 \times 29{,}000 \times 1200}$$ Converting unit ft → in

$$= \underline{\underline{1.034 \text{ in}}} \downarrow$$ Ans.

4) Compute slope θ_P, the P system is unchanged, while this time, we apply a dummy moment at point C (<u>Q system</u>).



$M = -1 \ (0 \le x \le 20)$

$\because \sum Q_M \theta_C = \int M_Q \frac{M_P dx}{EI}$

$$1 \cdot \theta_P = \int_0^8 (-1)\frac{(-0.6x^2-6x)}{EI}dx + \int_8^{20}(-1)\frac{(-0.6x^2-6x)}{2EI}dx$$

$$= \frac{1}{EI}\int_0^8 (0.6x^2+6x)dx + \int_8^{20}(0.6x^2+6x)dx \Big/ 2EI$$

$$= \frac{1}{EI}(0.2x^3+3x^2)\Big|_0^8 + \frac{1}{2EI}(0.2x^3+3x^2)\Big|_8^{20}$$

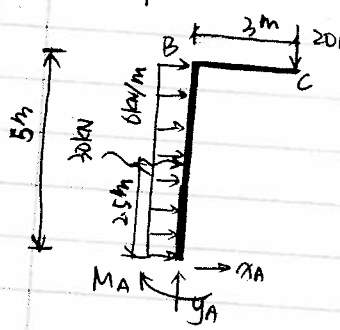
$$= \frac{294.4}{EI} + \frac{2505.6}{2EI}$$

$$= \frac{294.4\times(12^2)}{29{,}000\times 1200} + \frac{2505.6\times(12^2)}{2\times 29{,}000\times 1200}$$ converting unit ft → in

$= \underline{\underline{6.4\times10^{-3}\text{rad}}}$ Ans.

P10.26 Compute vertical and horizontal displacement at point C. Given $E = 200GPa$, $I = 240\times10^6 mm^4$.

1) Compute external reactions (P system) $\uparrow \rightarrow \circlearrowright$ (+)

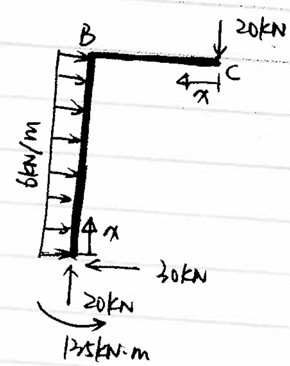


$\sum F(x) = x_A + 30kN = 0 \qquad x_A = -30kN \leftarrow$

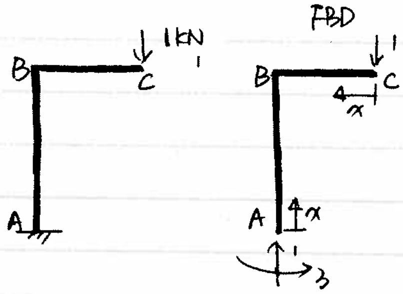
$\sum M_A = M_A + 30\times2.5 + 20\times3 = 0 \qquad M_A = -135kN\cdot m$ ↻

$\sum F_y = y_A - 20 = 0 \qquad y_A = 20kN \uparrow$

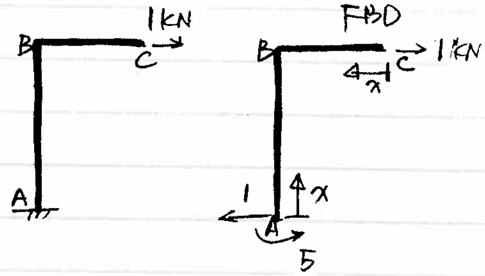
Draw FBD of the frame:



Apply dummy load 1kip horizontally / vertically to point C, thus we have two Q systems, compute external reactions based on these different loading conditions.



Q system for δ_{CV}

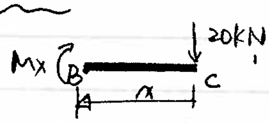


Q system for δ_{CH}

2) Write M in terms of x. (original point and direction for each member is illustrated in FBD)

P system

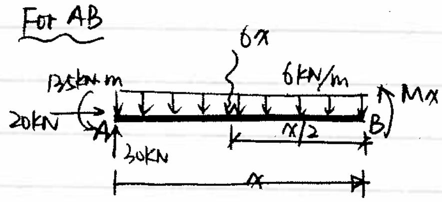
For BC



$\Sigma M_x = M_x + 20x = 0$

$M_x = -20x \ (0 \le x \le 3m)$

For AB

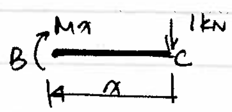


$\Sigma M_x = -135 + 30x - 6x \cdot \frac{1}{2}x - M_x = 0$

$M_x = 30x - 3x^2 - 135 \ (0 \le x \le 5m)$

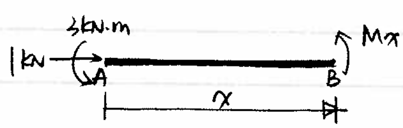
Q system for δ_{CV}

For BC



$\Sigma M_x = M_x + x = 0 \Rightarrow M_x = -x \ (0 \le x \le 3m)$

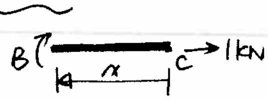
For AB



$\Sigma M_x = -3 - M_x = 0 \Rightarrow M_x = -3 \ kN \cdot m \ (0 \le x \le 5m)$

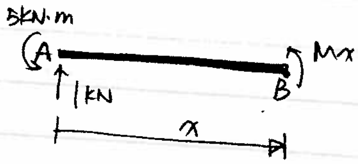
Q system for δ_{CH}

For BC



$M = 0 \ (0 \le x \le 3m)$

For AB



$\Sigma M_x = -5 + x - M_x = 0$

$\Rightarrow M_x = -5 + x = x - 5 \quad (0 \le x \le 5m)$

We can list these values in a diagram shown below.

Seg	Origin	range of x	M_P	M_Q for δ_{CV}	M_Q for δ_{CH}
AB	A	$0 \le x \le 5$	$30x - 3x^2 - 135$	-3	$x-5$
BC	C	$0 \le x \le 3$	$-20x$	$-x$	0

3) Compute displacement of point C.

$\because \Sigma Q \delta_P = \int M_Q \frac{M_P dx}{EI}$

$$\therefore 1kN \cdot \delta_{CV} = \int_{AB} M_Q \frac{M_P dx}{EI} + \int_{BC} M_Q \frac{M_P dx}{EI}$$

$$= \int_0^5 (-3) \frac{30x - 3x^2 - 135}{EI} dx + \int_0^3 (-x) \frac{-20x}{EI} dx$$

$$= \frac{1}{EI} \int_0^5 (9x^2 - 90x + 135 \times 3) dx + \frac{1}{EI} \int_0^3 20x^2 dx$$

$$= \frac{1}{EI} (3x^3 - 45x^2 + 405x) \Big|_0^5 + \frac{1}{EI} \left(\frac{20}{3} x^3\right) \Big|_0^3$$

$$= \frac{+1275}{EI} + \frac{180}{EI} = \frac{1455}{EI}$$

$$= \frac{1455 \times 10^3}{200 \times 10^9 \frac{N}{m^2} \times 240 \times 10^6 mm^4 \times (10^{-3})^4}$$

10^3: kN → N; $(10^{-3})^4$: mm → m

$$= 0.0303 m$$

$$= \underline{\underline{30.3 mm}} \ (\downarrow) \text{ Ans.}$$

*. Pay attention to the units! Don't use mixed units!

$$1kN \cdot \delta_{CH} = \int_{AB} \frac{M_Q M_P dx}{EI} + \int_{BC} \frac{M_Q M_P dx}{EI}$$

$$= \int_0^5 (x-5) \frac{(30x - 135 - 3x^2)}{EI} dx + 0 = \frac{1}{EI} \int_0^5 (30x^2 - 3x^3 - 135x - 150x + 15x^2 + 675) dx$$

$$= \frac{1}{EI} \int_0^5 (-3x^3 + 45x^2 - 285x + 675) dx$$

$$= \frac{1}{EI} (-0.75x^4 + 15x^3 - 142.5x^2 + 675x) \Big|_0^5$$

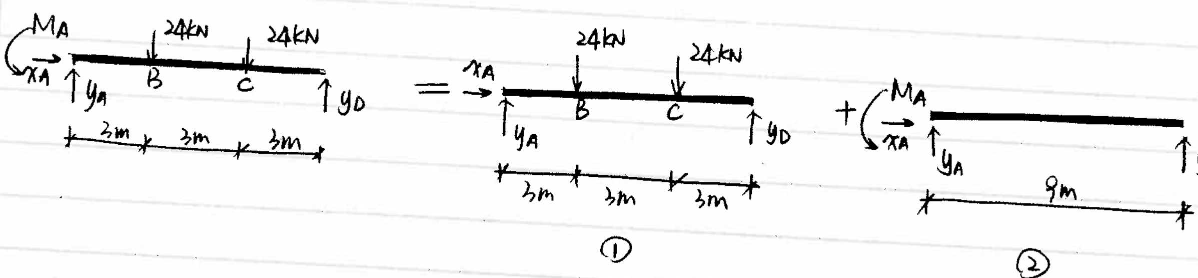
$$= \frac{1218.75}{EI} = \frac{1218.75 \times 10^3}{200 \times 10^9 \times 240 \times 10^6 \times (10^{-3})^4} = 0.0254 m = \underline{\underline{25.4 mm}} \rightarrow \text{ Ans.}$$

P10.28 Determine the value of moment must be applied to the left end of the beam if the slope at A is to be zero. EI is constant. Assume that rocker at support D acts as a roller.

Note! The moment expression is different within different ranges (AB, BC, CD), so we should integrate them seperately according to different ranges.

1). Compute external reactions (P system), then write M in terms of x. The original point & direction is illustrated in FBD.

Because of the principle of superposition, the combined loading condition could be divided into 2 parts, that is:

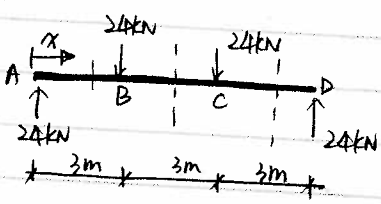


We can compute external reactions, write M expressions seperately, then add them together.

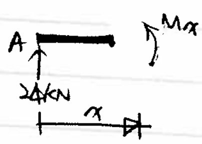
For ①

Symmetric loading condition, the reactions must be symmetric!

FBD



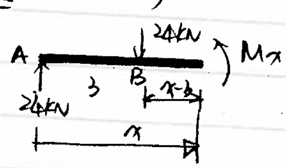
AB ($0 \le x \le 3m$)



$24x - M_x = 0$

$M_x = 24x \quad (0 \le x \le 3m)$

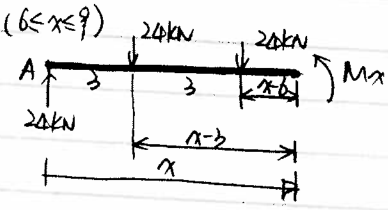
BC ($3 \le x \le 6$)



$\Sigma M_x = 24x - 24(x-3) - M_x = 0$

$M_x = 72 \text{ kN}\cdot\text{m} \quad (3 \le x \le 6)$

CD ($6 \le x \le 9$)



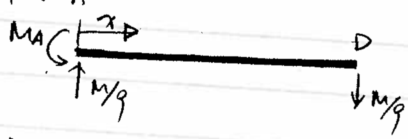
$\Sigma M_x = 24x - 24(x-3) - 24(x-6) - M_x = 0$

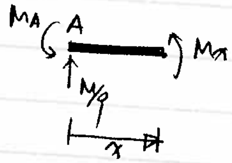
$\Rightarrow M_x = 24x - 24(x-3) - 24(x-6)$

$= -24x + 216 \quad (6 \le x \le 9)$

For loading condition ②

FBD A





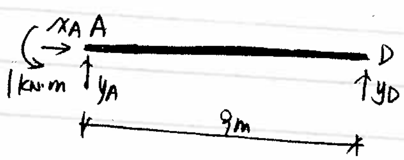
$\Sigma M_x = -M_A + M_A/9 \cdot x - M_x = 0$

$\Rightarrow M_x = M_A/9 \cdot x - M_A \quad (0 \le x \le 9m)$

2) If we want to compute slope at A, we should apply a dummy moment at point A, then compute external reactions under this loading condition, write M in terms of x.

↺ ⊕

Q system

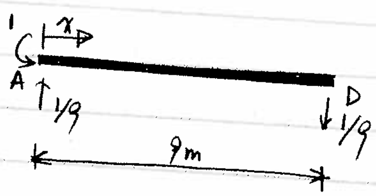


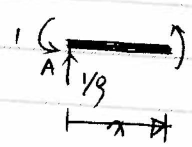
$\Sigma F(x) = x_A = 0$

$\Sigma M_A = -1 - y_D \times 9 = 0 \quad y_D = -1/9 \downarrow$

$\Sigma F_y = y_A + y_D = 0 \Rightarrow y_A = 1/9 \uparrow$

FBD





$\Sigma M_x = -1 + 1/9 \cdot x - M_x = 0$

$\Rightarrow M_x = x/9 - 1 \quad (0 \le x \le 9m)$

We can list these values in the diagram shown below:

Seg	Origin	range of x	M_Q	$M_{P①}$	$M_{P②}$
AB		$0 \le x \le 3$		$24x$	
BC	A	$3 \le x \le 6$	$\frac{x}{9} - 1$	72	$\frac{M_A}{9} \cdot x - M_A$
CD		$6 \le x \le 9$		$-24x + 216$	

3) Compute the slope, then set it to be zero, get M_A value.

$\because \Sigma Q_m \theta_p = \int M_Q \frac{M_p\,dx}{EI} = \int M_Q \frac{(M_{p1}+M_{p2})\,dx}{EI}$

$\therefore 1\,kN\cdot m\ \theta_p = \int_{AB}(\quad) + \int_{BC}(\quad) + \int_{CD}(\quad)$

$= \int_0^3 \left(\frac{x}{9}-1\right)\frac{(24x+M_{p2})}{EI}dx + \int_3^6 \left(\frac{x}{9}-1\right)\frac{(72+M_{p2})}{EI}dx + \int_6^9 \left(\frac{x}{9}-1\right)\frac{(-24x+216+M_{p2})}{EI}dx$

$= \int_0^3 \left(\frac{x}{9}-1\right)\frac{24x}{EI}dx + \int_3^6 \left(\frac{x}{9}-1\right)\frac{72}{EI}dx + \int_6^9 \left(\frac{x}{9}-1\right)\frac{(-24x+216)}{EI}dx + \int_0^9 \left(\frac{x}{9}-1\right)\frac{M_{p2}}{EI}dx$

$= 0$

$\Rightarrow$ cancel out EI

$\int_0^3 \left(\frac{x}{9}-1\right)24x\,dx + \int_3^6 \left(\frac{x}{9}-1\right)72\,dx + \int_6^9 \left(\frac{x}{9}-1\right)(-24x+216)\,dx + \int_0^9 \left(\frac{x}{9}-1\right)\left(\frac{M_A}{9}x - M_A\right)dx$

$\Rightarrow \int_0^3 \left(\frac{8}{3}x^2-24x\right)dx + \int_3^6 (8x-72)\,dx + \int_6^9 \left(-\frac{8}{3}x^2+24x+24x-216\right)dx + \int_0^9 \left(\frac{M_A}{81}x^2 - \frac{2M_A}{9}x + M_A\right)dx$

$\Rightarrow \int_0^3 \left(\frac{8}{3}x^2-24x\right)dx + \int_3^6 (8x-72)\,dx + \int_6^9 \left(-\frac{8}{3}x^2+48x-216\right)dx + \int_0^9 \left(\frac{M_A}{81}x^2 - \frac{2M_A}{9}x + M_A\right)dx$

$\Rightarrow \left(\frac{8}{9}x^3-12x^2\right)\Big|_0^3 + \left(4x^2-72x\right)\Big|_3^6 + \left(-\frac{8}{9}x^3+24x^2-216x\right)\Big|_6^9 + \left(\frac{M_A}{243}x^3 - \frac{M_A}{9}x^2 + M_A x\right)\Big|_0^9$

$\Rightarrow (-84)+(-108)+(-24)+(3M_A)=0$

$\Rightarrow \underline{\underline{M_A = 72\,kN\cdot m}}$ ↺

Ans.